ISSN 2347-1921

Relationship between Path and Series Representations for the Three Basic Univalent G-functions

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ABSTRACT

In this paper we demonstrate how series representation for the three basicunivalent *G*-functions, namely $G_{0,2}^{1,0}$, $G_{1,2}^{1,1}$, and $G_{1,1}^{1,1}$ can be obtained from theirMellin-Barnes path integral representations.In two special cases, the images of thirdbasic univalent G-function $G_{1,1}^{1,1}$ are derived by the Biernacki and Libera operators.

Keywords: Meijer's G-function; Univalent function; Univalent G-function; Biernackioperator; Libera operator.

Council for Innovative Research

Peer Review Research Publishing System

Journal: Journal of Advances in Mathematics

Vol 6, No 1

editor@cirworld.com

www.cirworld.com, member.cirworld.com

ISSN 2347-1921

INTRODUCTION

In mathematics, the Meijer's *G*-function was introduced by Cornelis Simon Meijer(1946) as a very general function intended to include all elementary functions andmost of the known special functions, for instance:

- $\sin z = \sqrt{\pi} G^{1,0}_{0,2}(\frac{z^2}{4}|^{-}_{\frac{1}{2},0}), \qquad -\frac{\pi}{2} < \arg z \leq \frac{\pi}{2}$
- $\cos z = \sqrt{\pi} G^{1,0}_{0,2}(\frac{z^2}{4}|^{-}_{0,\frac{1}{2}}), \qquad \forall z$
- $\ln z = G^{1,2}_{2,2}(z-1|^{1,1}_{1,0}), \qquad \forall z$
- $J_\nu(z) = G^{1,0}_{0,2}(\frac{z^2}{4}|^{-}_{\frac{\nu}{2},\frac{-\nu}{2}}), \qquad -\frac{\pi}{2} < \arg z \leq \frac{\pi}{2}$

A definition of the Meijer's *G*-function is given by the path integral in the complexplane, called Mellin-Barnes type integral see [1-3]:

$$G^{m,n}_{p,q}\left({}^{a_1 \dots a_p}_{b_1 \dots b_q}|z\right) = \frac{1}{2\Pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)} z^s ds. \tag{1.1}$$

For the function

$$G^{m,n}_{p,q}\left({}^{a_1,\dots,a_p}_{b_1,\dots,b_q}|z\right) \tag{1.2}$$

The integers *m; n; p; q* are called orders of the *G*-function; $\mathbf{a_p}$ and $\mathbf{b_q}$ are called "parameters"and in general, they are complex numbers. The definition holds under theassumptions: $0 \leq m \leq q$ and $0 \leq n \leq p$, where *m, n, p,* and *q* are integer numbers. In[4] the univalent Meijer's *G*-functions are classified into three types in the form of thefollowing proposition:

Proposition 1.1*All of the univalent Meijer's G-functions,* $G^{1,p}_{p,q+1}$, *can be considered as the generalised (q-tuple) fractional differ-integrals of one of the three simplest univalent G-functions, namely,* $G^{1,0}_{0,2}$; $G^{1,1}_{1,2}$; *and* $G^{1,1}_{1,1}$, *depending on whether p < q; p = q; p = q+1*.

In [5-8] *G*-functions are directly obtained as the solution in Micro- and Nano-structures,and in physical models such as Diffusion equation, Laplace's equation, and Schrodingerequation, respectively.

The classical Erdélyi-Koberoperators $I^{(\gamma_k),(\delta_k)}_{1,m}$;*m* transform one univalent Meijers*G*-function of the lower rank to another univalent Meijers*G*-function of the upper rankas the following lemma:

Lemma 1.2 *Let* $|z| < 1$ ($|z| < 1$ *for* $p = q + 1$), *then*

$$G^{1,p}_{p,q+1}\left({}^{1-a_1,\dots,1-a_p}_{0,1-b_1,\dots,1-b_q}|-z\right) = \begin{cases} I^{a_p-1,b_q-a_p}_{1,1}\left\{G^{1,p-1}_{p-1,q}\left({}^{1-a_1,\dots,1-a_{p-1}}_{0,1-b_1,\dots,1-b_{q-1}}|-z\right)\right\} & \text{if} \quad b_q > a_p \\ D^{b_q-1,a_p-b_q}_{1,1}\left\{G^{1,p-1}_{p-1,q}\left({}^{1-a_1,\dots,1-a_{p-1}}_{0,1-b_1,\dots,1-b_{q-1}}|-z\right)\right\} & \text{if} \quad b_q < a_p \end{cases} \tag{1.3}$$

Under the following conditions:

$$\delta_k > 0, \mu = 1, \gamma_k > -2, \tag{1.4}$$

these operators in the space of analytic functions, A, maps the class A onto itself.

$$I^{(\gamma_k),(\delta_k)}_{1,m} f(z) = z\sum_{n=0}^{\infty} a_n \prod_{k=1}^{m} \frac{\Gamma(\gamma_k+n+2)}{\Gamma(\gamma_k+\delta_k+n+2)} z^n \tag{1.5}$$

maps the class A onto itself [9]. In [10] Kiryakova et al. introduced other form of definitions for well-known operators by using generalised fractional calculus. For instance form of the Biernacki and Libera operators are respectively as follows:

$$Bf(z) = \int_0^z \frac{f(\xi)}{\xi} d\xi = \int_0^1 \frac{f(z\sigma)}{\sigma} d\sigma = I_1^{-1,1} f(z), \tag{1.6}$$

and

$$Lf(z) = \frac{2}{z} \int_0^z f(\xi) d\xi = 2 \int_0^1 f(z\sigma) d\sigma = 2I_1^{0,1} f(z). \tag{1.7}$$

2 MAIN RESULTS

2.1The Firstbasicunivalent G-function

In the study of basic univalent *G*-functions we first determine the position of poles andzeroes of functions inside the path integral.

$$G_{0,2}^{1,0}[{}^{-}_{b_1,b_2}|z] = \frac{1}{2\pi i} \int_L \frac{\Gamma(b_1 - s) z^s}{\Gamma(1 - b_2 + s)} ds. \tag{2.1}$$

Here we see that:

1. Position of poles $\Gamma(b_1\text{-}s)$: $s = b_1 + n$; $n = 0,1,2,\ldots$.
2. Position of zeroes $\frac{1}{\Gamma(1-b_2+s)}$:$s = b_2\text{-}1\text{-}n$; $n = 0,1,2,\ldots$.

We obtain the following

Theorem 2.3*If L in (2.1) is a loop beginning and ending at +∞, encircling all polesof $\Gamma(b_1\text{-}s)$ exactly once in the negative direction, then*

$$G_{0,2}^{1,0}[{}^{-}_{b_1,b_2}|z] = \sum_{n=0}^{\infty} \frac{1}{\Gamma(1 + b_1 - b_2 + n)} \frac{z^{b_1+n}}{n!}. \tag{2.2}$$

Proof 1 At a simple pole, the residue of function *f* is given by

$$Res(f, c) = lim_{s\to c}(s - c) f(s).$$

So the residueis given by $\frac{(-1)^{n-1}}{n!}$. Then by putting $s = n + b_1$ in expression$\frac{z^s}{\Gamma(1-b_2+s)}$ we obtain (2.2).

Example 2.1 If $b_1 = 0$; $b_2 = 1/2$ and $z \to \frac{z^2}{4}$ in (2.1), then we get

$$\cos z = G_{0,2}^{1,0}[{}^{-}_{0,\frac{1}{2}}|\frac{z^2}{4}] = \sqrt{\pi}(\frac{1}{2\pi i}) \int \frac{\Gamma(-s) z^{2s}}{4\Gamma(\frac{1}{2} + s)} ds. \tag{2.3}$$

Corollary 2.4*Putting $b_1 = 0$; $b_2 = 1/2$ and $z \to \frac{z^2}{4}$ in (2.2) gives*

$$\cos z = G_{0,2}^{1,0}[{}^{-}_{0,\frac{1}{2}}|\frac{z^2}{4}] = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\frac{1}{2} + n)} \frac{z^{2n}}{4^n n!}. \tag{2.4}$$

$$while\ \Gamma(\tfrac{1}{2} + n) = \tfrac{(2n)!}{4^n n!}\sqrt{\pi} = \tfrac{(2n-1)!!}{2^n}\sqrt{\pi}\ we\ obtain$$

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$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}. \tag{2.5}$$

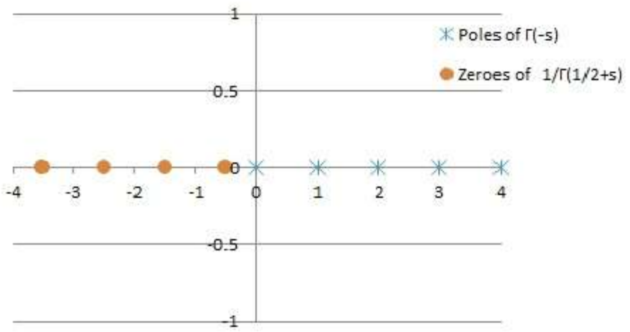

Figure 1: Poles and zeroes of cos z = $G^{1,0}_{0,2}\left[{- \atop 0,\frac{1}{2}}\middle|\frac{z^2}{4}\right]$

It is noted that the product of all the odd integers up to some odd positive integer *k*iscalled the double factorial of *k*, and denoted by *k!!* . Next,

Theorem 2.5 *Let p=1,q=2 in Lemma 1.2 such that*

$$G^{1,1}_{1,3}\left[{a_1 \atop 0,b_2,b_3}\middle| -z\right] = I_{1,1}^{-a_1,a_1-b_3} G^{1,0}_{0,2}\left[{- \atop 0,b_2}\middle| -z\right],$$

then (2.2) implies that

$$G^{1,1}_{1,3}\left[{a_1 \atop 0,b_2,b_3}\middle| -z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(1-a_1+n)}{\Gamma(1-b_2+n)\Gamma(1-b_3+n)} \frac{z^n}{n!}. \tag{2.6}$$

2.2 The secondbasicunivalent G-function

$$G^{1,1}_{1,2}\left[{a_1 \atop b_1,b_2}\middle| z\right] = \frac{1}{2\pi i} \int_L \frac{\Gamma(b_1-s)\Gamma(1-a_1+s)z^s}{\Gamma(1-b_2+s)} ds. \tag{2.7}$$

Here we see that:

1. Position of poles $\Gamma(b_1\text{-}s)$: $s = b_1 + n$; $n = 0,1,2,\ldots$.
2. Position of poles $\Gamma(1\text{-}a_1\text{+}s)$: $s = a_1\text{-}1\text{-}n$; $n = 0,1,2,\ldots$.
3. Position of zeroes $\frac{1}{\Gamma(1-b_2+s)}$:$s = b_2\text{-}1\text{-}n$; $n = 0,1,2,\ldots$.

Here we have

Theorem 2.6 *If a_1-b_1≠ 1,2,3,… , which implies that no pole of $\Gamma(b_1\text{-}s)$ coincideswith any pole of $\Gamma(1\text{-}a_1\text{+}s)$, then*

$$G^{1,1}_{1,2}\left[{a_1 \atop b_1,b_2}\middle| z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(1-a_1+b_1+n)}{\Gamma(1+b_1-b_2+n)} \frac{z^{b_1+n}}{n!}. \tag{2.8}$$

Proof 2. At a simple pole, the residue of function **f** is given by

ISSN 2347-1921

$$Res(f,c) = lim_{s\to c}(s-c)f(s).$$

L in (2.7) is a loop beginning and ending at $+\infty$, encircling all poles of $\Gamma(b_1\text{-}s)$ exactlyonce in the negative direction, but not encircling any pole of $\Gamma(1\text{-}a_1\text{+}s)$. So the residueis given by $\frac{(-1)^{n-1}}{n!}$. Then by putting s = n + 1 in $\frac{\Gamma(1-a1+s)z^s}{\Gamma(1-b_2+s)}$ we obtain (2:8).

Example 2.2 If $a_1 = 1 + I$, $b_1 = 0$ and $b_2 = -i + 1/2$ in (2.7), then we have

$$G^{1,1}_{1,2}\left[{}^{1+i}_{0,i+\frac{1}{2}}|z\right] = \frac{1}{2\pi i}\int_L \frac{\Gamma(-s)\Gamma(-i+s)}{\Gamma(\frac{1}{2}+i+s)}z^s ds.$$

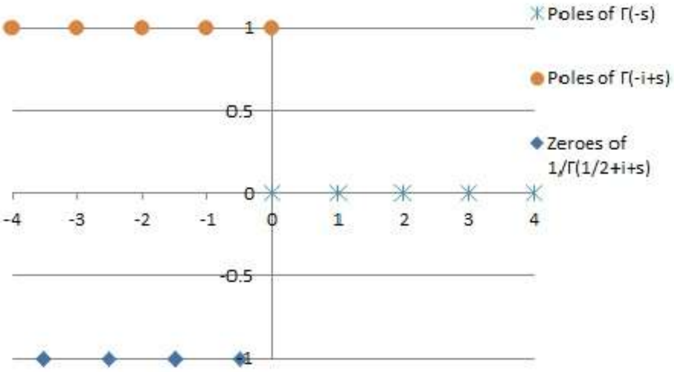

Figure 2: Poles and zeroes of $G^{1,1}_{1,2}\left[{}^{1+i}_{0,i+\frac{1}{2}}|z\right]$

If equal parameters appear among the a_p and b_q determining the factors in the numerator and the denominator of the integrand, the fraction can be simplified, and the order of the function thereby be reduced. If $a_1 = b_2$ then

$$G^{1,1}_{1,2}\left[{}^{a_1}_{b_1,b_2}|z\right] = G^{1,0}_{0,1}\left[{}^{-}_{b_1}|z\right] = \frac{1}{2\pi i}\int_L \Gamma(b_1-s)z^s ds. \qquad (2.9)$$

Here we see that:

1. Position of poles $\Gamma(b_1\text{-}s)$: $s = b_1 + n$; n = 0,1, 2,… .

2. Position of zeroes: there are no zeroes.

Example 2.3 If we put $b_1 = 0$ in (2.9); then we get exponential function

$$e^{-z} = G^{1,0}_{0,1}[0|z] = \frac{1}{2\pi i}\int_L \Gamma(-s)z^s ds. \qquad (2.10)$$

Corollary 2.7 Putting $a_1 = b_2$ and $b_1 = 0$ in (2.8) verifies (2.10)

$$e^{-z} = G^{1,0}_{0,1}[0|z] = \sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

Next an interesting result as follows:

Theorem 2.8 *Let p = 2, q = 2 in Lemma 1.2 such that*

$$G^{1,2}_{2,3}\left[{}^{a_1,a_2}_{0,b_2,b_3}|-z\right] = I^{-a_2,a_2-b_3}_{1,1}G^{1,1}_{1,2}\left[{}^{a_1}_{0,b_2}|-z\right],$$

then (2.8) implies that

$$G^{1,2}_{2,3}\left[{}^{a_1,a_2}_{0,b_2,b_3}\middle| -z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(1-a_1+n)\Gamma(1-a_2+n)}{\Gamma(1-b_2+n)\Gamma(1-b_3+n)} \frac{z^n}{n!} \tag{2.11}$$

Proof 4 Using (1.5) with $\gamma_k = -a_2$; $\delta_k = a_2 - b_3$; $m = 1$and$f(z) = G^{1,1}_{1,2}\left[{}^{a_1}_{b_1,b_2}|z\right]$givenby (2.8) yields series representation for $G^{1,2}_{2,3}\left[{}^{a_1,a_2}_{0,b_2,b_3}| -z\right]$.

2.3 The third basic univalent G-function

We begin with the definition of third basic univalentG-function, $G^{1,1}_{1,1}$,asfollows:

$$G^{1,1}_{1,1}\left[{}^{a_1}_{b_1}\middle| -z\right] = \frac{1}{2\pi i}\int_L \Gamma(b_1-s)\Gamma(1-a_1+s)(-z)^s ds. \tag{2.12}$$

Here we see that:

1. Position of poles$\Gamma(b_1$-s) : s=b_1+n; n=0,1,2,… .
2. Position of poles$\Gamma(1$-a_1+s) : s=a_1-1-n; n=0,1,2,… .
3. Position of zeroes: there are no zeroes.

Theorem 2.9 Let a_1-b_1≠1,2,3,…, which implies that no pole of Γ(b_1-s) coincides with any pole of Γ(1-a_1+s), then

$$G^{1,1}_{1,1}\left[{}^{a_1}_{b_1}\middle| -z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(1-a_1+b_1+n)}{n!} z^{b_1+n} \tag{2.13}$$

Proof 5 *At a simple pole, the residue of function **f** is given by*

$$Res(f,c) = lim_{s\to c}(s-c)f(s).$$

L in (2.12) is a loop beginning and ending at +∞, encircling all poles of Γ(b_1-s) exactly once in the negative direction, but not encircling any pole of Γ(1-a_1+s). So the residue is given with $\frac{(-1)^{n-1}}{n!}$. Then by putting s=n+1 in Γ(1-a_1+s)(-z)^s we obtain (2.13).

Example 2.4 If we puta_1= 0andb_1 = 1then the Koebe function can be obtained

$$K(z) = \frac{z}{(1-z)^2} = G^{1,1}_{1,1}\left[{}^{0}_{1}\middle| -z\right] = \frac{1}{2\pi i}\int_L \Gamma(1-s)\Gamma(1+s)z^s ds. \tag{2.14}$$

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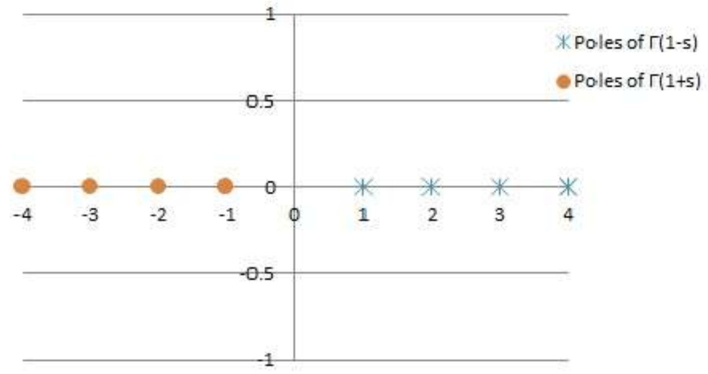

Figure 3: Poles of Koebe function $G^{1,1}_{1,1}[{}^{0}_{1} - z]$

Corollary 2.10 Putting $a_1 = 0$ and $b_1 = 1$ in (2.13) verifies (2.14)

$$G^{1,1}_{1,1}\left[{}^{0}_{1}\middle| -z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(2+n)}{n!} z^{n+1} = \sum_{n=0}^{\infty} (n+1) z^{n+1} = z + 2z^2 + 3z^3 + \ldots$$

Theorem 2.11 *Let p = 2; q = 1 in Lemma 1.2 such that*

$$G^{1,2}_{2,2}\left[{}^{a_1, a_2}_{0, b_2}\middle| -z\right] = I^{-a_2, a_2 - b_2}_{1,1} G^{1,1}_{1,1}\left[{}^{a_1}_{0}\middle| -z\right],$$

then (2.13) implies that

$$G^{1,2}_{2,2}\left[{}^{a_1, a_2}_{0, b_2}\middle| -z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(1 - a_1 + n)\Gamma(1 - a_2 + n)}{\Gamma(1 - b_2 + n)} \frac{z^n}{n!} \qquad (2.15)$$

Proof 6 Using (1.5) with $\gamma_k = -a_1;\ \delta_k = a_1 - b_2; m = 1$ and $f(z) = G^{1,1}_{1,1}[{}^{a_1}_{0}| - z]$givenby (2.13) yields series representation for $G^{1,2}_{2,2}[{}^{a_1,a_2}_{0,b_2}| - z]$.

Corollary 2.12 Putting $a_2 = 1$ and $b_2 = 0$ in (2.15), and using of (1.6) gives imageof $G^{1,1}_{1,1}$ by Biernacki operator

$$G^{1,2}_{2,2}\left[{}^{a_1, 1}_{0, 0}\middle| -z\right] = I^{-1,1}_{1,1} G^{1,1}_{1,1}\left[{}^{a_1}_{0}\middle| -z\right] = \sum_{n=0}^{\infty} \Gamma(1 - a_1 + n) \frac{z^n}{(n+1)!}.$$

Corollary 2.13 Putting $a_2 = 0$ and $b_2 = 0$ in (2.15), and using of (1.7) gives imageof$G^{1,1}_{1,1}$by Libera operator

$$G^{1,2}_{2,2}\left[{}^{a_1, 0}_{0, -1}\middle| -z\right] = I^{0,1}_{1,1} G^{1,1}_{1,1}\left[{}^{a_1}_{0}\middle| -z\right] = \sum_{n=0}^{\infty} \Gamma(1 - a_1 + n)(n+1) \frac{z^n}{(n+2)!}.$$

ACKNOWLEDGMENTS

This work was supported by MOHE with the grant number:ERGS/1/2013/STG06/UKM/01/2.

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